CNCM Online Round 3

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Problem 1

Problem

Harry, who is incredibly intellectual, needs to eat carrots C_1, C_2, C_3 and solve *Daily Challenge* problems D_1, D_2, D_3. However, he insists that carrot C_i must be eaten only after solving *Daily Challenge* problem D_i. In how many satisfactory orders can he complete all six actions?

Solution

There are $6! = 720$ possible permutations of $C_1, C_2, C_3, D_1, D_2, D_3$. However, only looking at D_1 and C_1, there exists exactly one permutation (out of the two possible) that makes sense - namely,

$$D_1 \quad C_1$$

Thus, there is a $\frac{1}{2}$ chance this happens. The same thing happens with D_2, D_3, C_2, C_3, so thus we get that only $\frac{1}{8}$ of the permutations will work. This gives

$$720 \cdot \frac{1}{8} = \boxed{\textbf{90}}$$

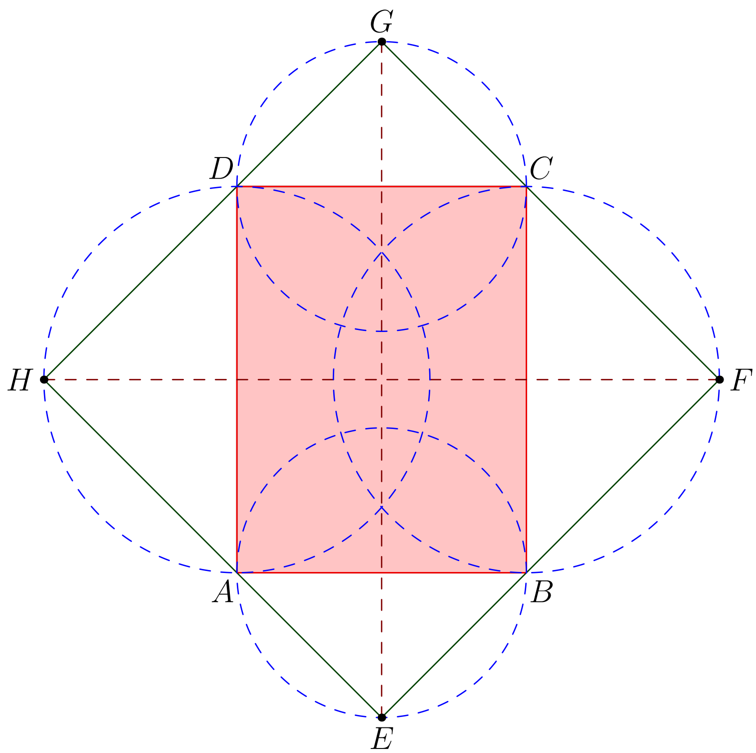
such permutations.

Problem 2

Problem

Consider rectangle $ABCD$ with $AB = 6$ and $BC = 8$. Pick points E, F, G, H such that the angles $\angle AEB, \angle BFC, \angle CGD, \angle AHD$ are all right. What is the largest possible area of quadrilateral $EFGH$?

Solution



Note that the condition $\angle AEB = 90^\circ$ means that E lies on the circle diameter AB, and similarly with F, G, H. Thus we can get

$$[EFGH] = \frac{EG \cdot FH \cdot \sin\angle(EG, FH)}{2}$$

where $\angle(EG, FH)$ is the angle formed by those lines (does not matter which one, as $\sin x = \sin 180^\circ - x$). However, we know that $EG \leq 3 + 3 + 8$ and $FH \leq 4 + 4 + 6$, so thus

$$[EFGH] \leq \frac{14 \cdot 14 \cdot 1}{2} = \boxed{\mathbf{98}}$$

Equality is achievable when AEB, BFC, CGD, DHA are all isosceles, or $EFGH$ is a square.

Problem 3

Problem

Let $a_1 = 1$ and $a_{n+1} = a_n \cdot p_n$ for $n \geq 1$ where p_n is the nth prime number, starting with $p_1 = 2$. Let $\tau(x)$ be equal to the number of divisors of x. Find the remainder when

$$\sum_{n=1}^{2020} \sum_{d|a_n} \tau(d)$$

is divided by 91 for positive integers d. Recall that $d \mid a_n$ denotes that d divides a_n.

Solution

We can note that this is very vacuously defined - for any $n \geq 2$, we have

$$a_n = \prod_{k=1}^{n-1} p_k$$

where p_i the sequence of increasing primes, as in the problem. Thus, we get a_n is squarefree, so it has 2^n divisors. Now, consider some divisor of a_n with k prime factors. Then, there are a total of $\binom{n-1}{k}$ possible choices for that divisor, which has 2^k divisors. Thus, we get

$$\sum_{d|a_n} \tau(d) = \sum_{k=0}^{n-1} \binom{n-1}{k} 2^k = 3^{n-1}$$

by the binomial formula. Thus, we get

$$\sum_{n=1}^{2020} \sum_{d|a_n} \tau(d) = \sum_{n=1}^{2020} 3^{n-1} = \frac{3^{2020} - 1}{3 - 1}$$

Note that $\varphi(91) = 6 \cdot 12 = 72$, and thus we get

$$3^{2016} \equiv \left(3^{72}\right)^{28} \equiv 1^{28} \equiv 1 \pmod{91}$$

so thus the answer is equivalent (mod 91) to

$$\frac{3^{2020} - 1}{3 - 1} \equiv \frac{3^4 - 1}{3 - 1} = \boxed{40}$$

Problem 4

Problem

Hari is obsessed with cubics. He comes up with a cubic with leading coefficient 1, rational coefficients and real roots $0 < a < b < c < 1$. He knows the following three facts: $P(0) = -\frac{1}{8}$, the roots for a geometric progression in the order a, b, c, and

$$\sum_{k=1}^{\infty}(a^k + b^k + c^k) = \frac{9}{2}$$

The value $a + b + c$ can be expressed as $\frac{m}{n}$, where m, n are relatively prime positive integers. Find $m + n$.

Solution

We note that $-\frac{1}{8} = P(0) = (0 - a)(0 - b)(0 - c) = -abc = -b^3$, so thus $b = \frac{1}{2}$. Now, we note that

$$\sum_{k=1}^{\infty} a^k = \frac{a}{1-a}$$

and similarly with b, c. Thus, we get that

$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} = \frac{9}{2}$$

Then, we get that $ac = \frac{1}{4}$, so thus $c = \frac{1}{4a}$. Thus, we get

$$\frac{a}{1-a} + 1 + \frac{1/4a}{1-1/4a} = \frac{a}{1-a} + 1 + \frac{1}{4a-1} = \frac{9}{2}$$

Thus, we get that

$$\frac{4a^2 - 2a + 1}{-4a^2 + 5a - 1} = \frac{7}{2}$$

or equivalently

$$8a^2 - 4a + 2 = -28a^2 + 35a - 7$$

It's possible to solve this to get $a = \frac{1}{3}, \frac{3}{4}$. However, we get that by Vieta's formulas that the sum of the two possible values of a are $\frac{13}{12}$, making $a + b + c = \frac{19}{12}$. Thus, the sum is $\boxed{\textbf{31}}$.

Problem 5

Problem

How many positive integers N less than 10^{1000} are such that N has x digits when written in base ten and $\frac{1}{N}$ has x digits after the decimal point when written in base ten? For example, 20 has two digits and $\frac{1}{20} = 0.05$ has two digits after the decimal point, so 20 is a valid N.

Solution

We note that if N works, so does $10N$. Thus, it suffices to check powers of 2 and 5 (the decimals have to terminate, clearly).

We note that $\frac{1}{2^k}$ has exactly k digits after the decimal point, so we need 2^k to have exactly k digits. However, we can verify the inequality

$$2^k < 10^{k-1}$$

for $k \geq 2$, with the case $k = 2$ being clear and then inducting on k. Thus, only 2 works (note 1 trivially fails). Similarly, if we do the exact same thing with 5^k, we get the inequality

$$5^k < 10^{k-1}$$

for $k \geq 4$. Thus, the only powers of 5 that can work are $5, 25, 125$. However, we can multiply by powers of 10 - we can multiply 2 by any power of 10 from 10^0 to 10^{1000}. We can do the same thing to 5. For 25, the powers of 10 are 10^0 to 10^{999}, while for 125, the valid powers of 10 are from 10^0 to 10^{998}. Thus the answer is $1000 + 1000 + 999 + 998 = \boxed{\textbf{3997}}$.

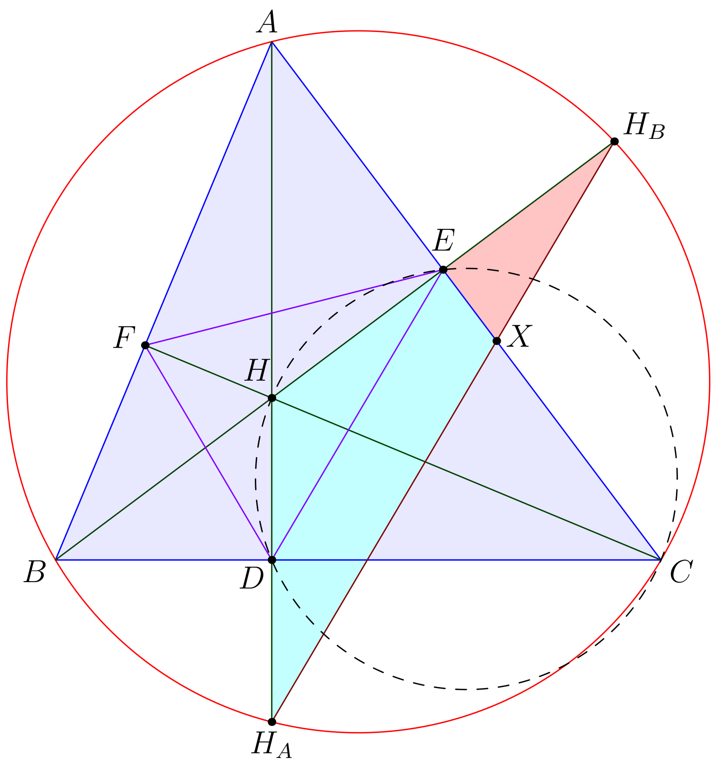
Problem 6

Problem

Triangle ABC has side lengths $AB = 13, BC = 14$, and $CA = 15$. Let Γ denote the circumcircle of $\triangle ABC$. Let H be the orthocenter of $\triangle ABC$. Let AH intersect Γ at a point D other than A. Let BH intersect AC at F and Γ at a point G other than B. Suppose DG intersects AC at X. Compute the greatest integer less than or equal to the area of quadrilateral $HDXF$.

Solution

We will rename some points. Let D be the foot from A to BC, E the foot from B to CA, and F the foot from C to AB. Let the intersection of line AHD with Γ be H_A, and intersection of line BHE with Γ be H_B. Let X be the intersection of H_AH_B and AC. Our goal is to compute the floor of the area of HH_AH_BX.



It is common knowledge that H_A is the reflection of H over D, and H_B the reflection of H over E. Indeed, note that

$$\measuredangle H_ABC = \measuredangle H_AAC = 90^\circ - \measuredangle ACB = \measuredangle CBH$$

implying the conclusion. The same angle chase resolves the scenario for H_B. Now, we can note that $EHDC$ is cyclic, so thus $\angle HED = \angle HCD = 90^\circ - \angle B$. However, by our reflection claim, we get that $\triangle HDE \sim \triangle HH_AH_B$ (by SAS similarity) and thus $\angle HH_BX = 90^\circ - \angle B$. Now, we move on to the computations.

First, we write down

$$[HH_AXE] = [HH_AH_B] - [H_BEX]$$

To compute the first area, we note that

$$[HH_AH_B] = \frac{HH_A \cdot HH_B \cdot \sin\angle H_BHH_A}{2} = 2HD \cdot HE \cdot \sin\angle C$$

We can compute that by Heron's Formula,

$$s = \frac{13+14+15}{2} = 21 \qquad [ABC] = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{8\cdot 7\cdot 6\cdot 21} = 84$$

Thus, we can get

$$\sin C = \frac{2[ABC]}{ab} = \frac{4}{5}$$
$$\sin B = \frac{2[ABC]}{ac} = \frac{12}{13}$$
$$\sin A = \frac{2[ABC]}{bc} = \frac{56}{65}$$

In addition, we get

$$BD = AB\cos\angle B = 13\cdot\frac{5}{13} = 5$$

and thus

$$HD = \frac{BD}{\tan\angle C} = \frac{5}{4/3} = \frac{15}{4}$$

Similarly, we can obtain

$$CE = BC\cos\angle C = 14\cdot\frac{3}{5} = \frac{42}{5}$$

and thus

$$HE = \frac{CE}{\tan\angle A} = \frac{42/5}{56/33} = \frac{99}{20}$$

Thus, we can finally calculate

$$[HH_AH_B] = 2\cdot\frac{99}{20}\cdot\frac{15}{4}\cdot\frac{4}{5} = \frac{297}{10}$$

Now, to calculate the last part, note that

$$[H_BEX] = \frac{H_BE\cdot EX}{2} = \frac{H_BE^2}{2\tan\angle B} = \frac{HE^2}{2\tan\angle B} = \frac{3267}{640}$$

Now, to finish off, we get that our answer is $24\frac{381}{640}$, giving the floor as $\boxed{\textbf{24}}$.

Problem 7

Problem

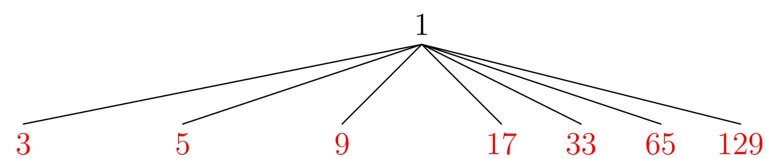
A subset of the nonnegative integers S is said to be a *configuration* if $200 \notin S$ and for all nonnegative integers x, $x \in S$ if and only if both $2x \in S$ and $\left\lfloor \frac{x}{2} \right\rfloor \in S$. Let the number of subsets of $\{1, 2, 3, \ldots, 130\}$ that are equal to the intersection of $\{1, 2, 3, \ldots, 130\}$ with some configuration S equal k. Compute the remainder when k is divided by 1810.

Solution

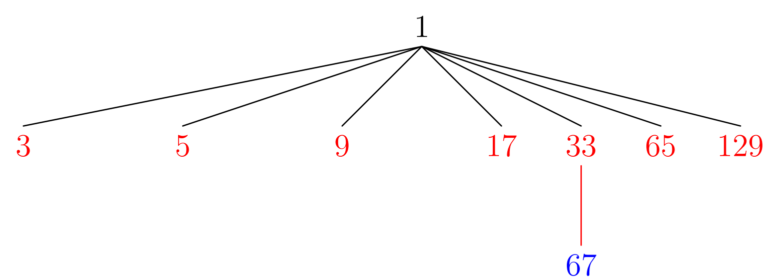
Evens are dumb, as if $2x \in S$, $x \in S$. So we only look at odd numbers. This also means we rewrite $200 \notin S$ as $25 \notin S$.

Now, we note that if we have an element in S, $1 \in S$ because repeatedly dividing by 2 will get you a 1. Now, we start with 1 and build up. We consider a rooted tree (root 1). But how does the tree work?

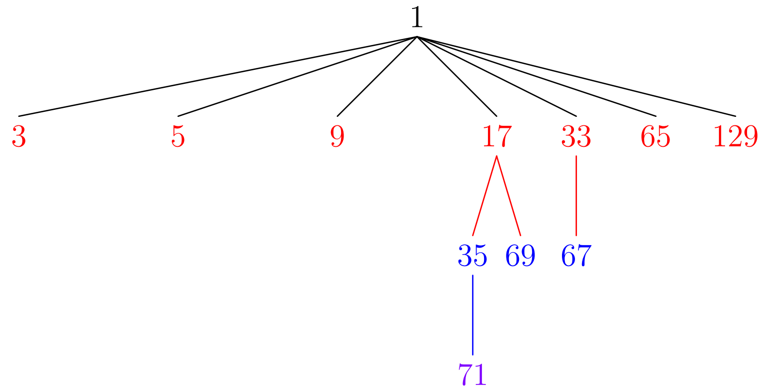
We actually consider how far away from 1 it is (how many +1 we have to do). Thinking of it as an option to either do $2x$ or $2x + 1$, we consider how many of the latter we perform. For example, the first two rows of the tree looks like:



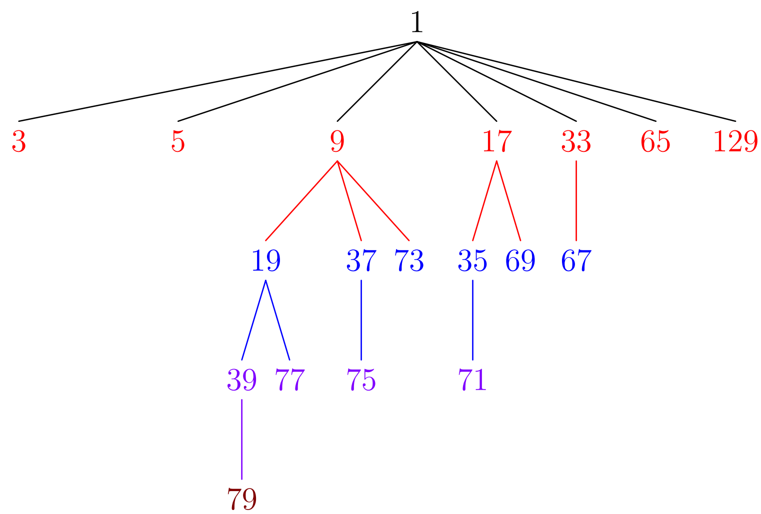
Now, let's count from right to left. 129 has 2 options - in or not. 65 has 2 options as well, as all its children are greater than 130. However, 33 has a child node:



Thus, there are 3 options: nothing, 33, and $33, 67$. Similarly, 17 has 2 children nodes:



35 has three options of where to belong, but 69 has only two (from our work on 33 and 65). Thus, we get $1 + 2 \cdot 3 = 7$ possible ways we can look at the 17 sub-branch. I'll draw the tree once more for 9 before stopping:



However, it's easy to see how this generalizes: 9 has

$$2 \cdot 3 \cdot 7 + 1 = 43$$

possible appearances, while 5 has

$$2 \cdot 3 \cdot 7 \cdot 43 + 1 = 1807 \equiv -3 \pmod{1810}$$

possible appearances. If we were careless, we would apply the same procedure to 3. However, we need to account for $25 \notin S$. We can note that $3 \cdot 8 + 1 = 25$, so 25 is a (direct) child of 3. In particular, 25 has the exact same child structure as 17, so we can instead say that for 3, we have

$$2 \cdot 3 \cdot 43 \cdot 1807 + 1 \equiv 258 \cdot (-3) + 1 \equiv 1037 \pmod{1810}$$

possible appearances. Thus, our answer is

$$2 \cdot 2 \cdot 3 \cdot 7 \cdot 43 \cdot 1807 \cdot 1037 + 1 \equiv 24 \cdot 1037 + 1 \equiv \boxed{\textbf{1359}} \pmod{1810}$$